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Integrable $Z_n \times Z_n$ Belavin model with non-trivial boundary terms

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Abstract. The open chain corresponding to the Belavin model is constructed by generalizing Skylanin's formalism to the case of the R-matrix with Z_n symmetry.

It is well known that quantum Yang–Baxter equation (QYBE) plays a key role in exactly solvable statistical models and integrable field theory. Recently, exact solutions of the QYBE have been studied fruitfully [1-5]. One way to study exactly solvable statistical systems is the quantum inverse scattering method (QISM) which was initiated by Faddeev and Takhtajan [6]. Skylyanin [7] had solved the open spin-$\frac{1}{2}H_{xxz}$ model by generalizing QISM to systems with independent boundary conditions on each end. This model with proper boundary conditions has the quantum group symmetry of $SU_q(2)$ [8, 10]. Therefore, Skylanin's method can be used to find new exactly solvable statistical models with quantum group symmetries. In Skylanin's paper [7], he assumes that R-matrices possess the following properties

$$P_{12}R_{12}(u)P_{12}=R_{12}(u) \tag{1}$$

$$R_{12}^{t_1}(u)=R_{12}^{t_2}(u) \tag{2}$$

$$R_{12}(u)R_{12}(-u)=\rho(u)\mathrm{id} \tag{3}$$

$$R_{12}^{t_1}(u)R_{12}^{t_2}(u-2\eta)=\tilde{\rho}(u)\mathrm{id} \tag{4}$$

where t_i denotes transposition in the ith space and id an identical operator. The $\rho(u)$ and $\tilde{\rho}(u)$ are some scalar functions. Unfortunately, most of the solutions of QYBE do not satisfy Sklyanin's assumption. Mezincescu and Nepomechie [9] extended Sklyanin's formalism to systems with the PT symmetric R-matrices. The restrictive conditions of this generalization are

$$P_{12}R_{12}(u)P_{12}=R_{12}^{t_1t_2}(u) \tag{5}$$

$$R_{12}(u)=\overset{1}{V}R_{12}^{t_2}(-u-\eta)\overset{1}{V}{}^{-1} \tag{6}$$

$$R_{12}(u)R_{12}^{t_1t_2}(-u)=\rho(u)\mathrm{id} \tag{7}$$

$$R_{12}^{t_1}\overset{1}{M}R_{12}^{t_2}(-u-2\eta)\overset{1}{M}{}^{-1}=\tilde{\rho}(u+\eta)\mathrm{id} \tag{8}$$

where $\overset{1}{V}$ stands for $V\otimes 1$, V is a matrix determined by R-matrix and $M=V^tV$. The condition (8) can be derived from (6) and (7). However, the R-matrix based on A_n^1 for $n>1$ does not have crossing symmetry (7). The spin open chains, which correspond to such R-matrices, cannot be treated directly using Sklyanin's formalism and its generalization.

Because the Z_n symmetric solution of the QYBE is related to algebra A_{n-1}^1, to exploit the symmetric properties of the Belavin $Z_n\times Z_n$ symmetric model is helpful for solving the above open problem. We have recently shown [11] that the Belavin solution R of QYBE satisfies the following symmetries

$$P_{12}R_{12}(u)P_{12}=R_{12}^{h_1h_2}(u) \tag{9}$$

$$R_{12}(u)R_{12}^{h_1h_2}(-u)=\rho(u)\mathrm{id} \tag{10}$$

$$R_{12}^{h_1}(u)R_{12}^{h_2}(-u-nw)=\tilde{\rho}(u,w)\mathrm{id}. \tag{11}$$

The superscript h_i denotes the Hermitian conjugation in the ith vector space and w is a new variable defined by $\eta=w/n+\frac{1}{2}+\tau/2$. It is obvious that the relations (9)–(11) are not equivalent with Sklyanin's assumption (1)–(4) and its generalization (5)–(8).

In this paper, we extend their formalisms to the case of the R-matrix satisfying (9)–(11) to find the Hamiltonian of the Belavin model with independent boundary conditions. Recently, Hou *et al* had shown [12] that the quantum group $SL_q(n)$ can be considered as a limit of the quantum symmetric algebra in the $Z_n\times Z_n$ Belavin model, which is the generalized Skylanin algebra [13]. Hence, the formalism developed in this paper can be used to construct the Hamiltonian of the spin chain with quantum group $SL_q(n)$ symmetry.

First of all, let us recall the fundamentals of the $Z_n\times Z_n$ Belavin model [2] and the major results in our paper [11].

The Boltzmann weight of the $Z_n\times Z_n$ Belavin model can be written as

$$R_{jk}(u)=\sum_{\alpha\in Z_n^2}W_\alpha(u)I_\alpha^{(j)}I_\alpha^{(k)\dagger} \tag{12}$$

where † stands for Hermitian conjugation and $I_\alpha^{(j)}$ acts on the subspace of the jth site, $I_\alpha=h^{\alpha_1}g^{\alpha_2}$, h and g are the $n\times n$ matrices with elements

$$h_{jk}=\delta_{j(\mathrm{mod}\,n)}^{k+1}\qquad g_{jk}=\omega^k\delta_{jk} \tag{13}$$

ω is equal to $\exp(i2\pi/n)$. The Boltzmann coordinate $W_\alpha(u)$ can be expressed in terms of the Jacobi theta function of rational characteristics $(\frac{1}{2}+\alpha_1/n,\frac{1}{2}+\alpha_2/n)$

$$\sigma_\alpha(u)\stackrel{\mathrm{def}}{=}\sum_{m=-\infty}^{\infty}\exp\left\{i\pi\tau\left(m+\frac{1}{2}+\frac{\alpha_1}{n}\right)^2+i2\pi\left(m+\frac{1}{2}+\frac{\alpha_1}{n}\right)\left(u+\frac{1}{2}+\frac{\alpha_2}{n}\right)\right\}. \tag{14}$$

$W_\alpha(u)$ is read as

$$W_\alpha(u)=\frac{\sigma_\alpha(u+\eta)\sigma_0(\eta)}{\sigma_\alpha(\eta)\sigma_0(u+\eta)}. \tag{15}$$

The Boltzmann weights satisfy the QYBE

$$R_{12}(u-v)R_{13}(u)R_{23}(v)=R_{23}(v)R_{13}(u)R_{12}(u-v). \tag{16}$$

The R-matrix of the Belavin model satisfies the symmetries (9)–(11), in which the explicit expressions of the scalar functions are

$$\rho(u)=n^2\frac{\theta\begin{bmatrix}\frac{1}{2}\\ \frac{1}{2}\end{bmatrix}(u+w,\tau)\,\theta\begin{bmatrix}\frac{1}{2}\\ \frac{1}{2}\end{bmatrix}(-u+w,\tau)}{\theta^2\begin{bmatrix}\frac{1}{2}\\ \frac{1}{2}\end{bmatrix}(w,\tau)} \tag{17}$$

and

$$\tilde{\rho}(u)=n^2\exp\{i\pi nw\}\frac{\theta\begin{bmatrix}\frac{1}{2}\\ \frac{1}{2}\end{bmatrix}(u,\tau)\,\theta\begin{bmatrix}\frac{1}{2}\\ \frac{1}{2}\end{bmatrix}(-u-nw,\tau)}{\theta^2\begin{bmatrix}\frac{1}{2}\\ \frac{1}{2}\end{bmatrix}(w,\tau)} \tag{18}$$

where

$$\theta\begin{bmatrix}\frac{1}{2}\\ \frac{1}{2}\end{bmatrix}(z,\tau)=\sigma_0(z).$$

In the operator representation, we can rewrite the QYBE (16) as

$$R_{12}(u-v)\overset{1}{L}(u)\overset{2}{L}(v)=\overset{2}{L}(v)\overset{1}{L}(u)R_{12}(u-v) \tag{19}$$

by introducing an operator

$$L(u)=\sum_{\alpha\in Z_n^2}W_\alpha(u)I_\alpha\otimes S_\alpha. \tag{20}$$

From equation (19), one can show that the quantum operator S_α is the operator of the generalized Sklyanin algebra [13].

In order to construct the Hamiltonian with independent boundary condition, we have to extend the Sklyanin formalism to the case of an R-matrix satisfying the restrictive conditions (9)-(11). We introduce two generalized algebras $\mathscr{T}_+$ and $\mathscr{T}_-$ which are defined by the following relations

$$R_{12}(u_-)\overset{1}{\mathscr{T}}_-(u_1)R_{12}^{h_1h_2}(u_+)\overset{2}{\mathscr{T}}_-(u_2)=\overset{2}{\mathscr{T}}_-(u_2)R_{12}(u_+)\overset{1}{\mathscr{T}}_-(u_1)R_{12}^{h_1h_2}(u_-) \tag{21}$$

and

$$\begin{aligned}R_{12}(-u_-)\overset{1}{\mathscr{T}}{}_+^{h_1}(u_1)R_{12}^{h_1h_2}(-u_+-nw)\overset{2}{\mathscr{T}}{}_+^{h_2}(u_2)\\ =\overset{2}{\mathscr{T}}{}_+^{h_2}(u_2)R_{12}(-u_+-nw)\overset{1}{\mathscr{T}}{}_+^{h_1}(u_1)R_{12}^{h_1h_2}(-u_-)\end{aligned} \tag{22}$$

where we have used the notation $u_\pm=u_1\pm u_2$. These algebras, especially $\mathscr{T}_-$, are the fundamental of our construction. Our goal is to find the solution of equations (21) and (22) for the R-matrix given by (12) and (15). Cherednik's work [14] gives an important hint to solve the problem. Define a matrix $\mathscr{K}(u)$ as

$$\mathscr{K}(u)=\frac{1}{n}\sum_{\alpha\in Z_n^2}W_{2\alpha}(u)\omega^{2\alpha_1\alpha_2}I_{2\alpha}. \tag{23}$$

The matrix $\mathscr{K}(u)$ satisfies the normalized condition $\mathscr{K}^2(0)=1$. Using the properties of the Jacobi theta function, one can show that $\mathscr{K}_-(u)=\mathscr{K}(u)\mathscr{K}(0)$ is a representation of the algebra $\mathscr{T}_-$ and the mapping

$$\phi:\mathscr{K}_-(u)\mapsto\mathscr{K}_+(u)=\mathscr{K}_-^h\left(-u-\frac{nw}{2}\right) \tag{24}$$

is isomorphic which gives a solution of equation (22). The proof of the above conclusions is a direct but rather tedious calculation. Here we only give the key steps of the proof. First, we substitute $\mathcal{K}_-(u)$ into equation (21) and taking use of the formulas (8) and (10) in [14]. Second, we take the Hermitian conjugation of equation (21) and replace u_i with $-u_i-nw/2$. The calculation shows that $\mathcal{K}_+(u)$ and $\mathcal{K}_-(u)$ defined as above give a representation of algebras $\mathcal{T}_+$ and $\mathcal{T}_-$ respectively. It is pointed out that the existence of $\mathcal{K}(u)$ means that of the solution of equations (21) and (22). If there exist inequivalent solutions, they correspond to spin chains with different boundary terms.

As usual, the monodromy matrix $T(u)$ is given by

$$T(u)=L_N(u)\ldots L_1(u) \tag{25}$$

where

$$L_j(u)=\sum_{\alpha\in Z_n^2} W_\alpha(u)I_\alpha S_\alpha^{(j)} \tag{26}$$

and the superscript j denotes the quantum space acted on by the operator S_α. Using the QYBE (19) and $\mathcal{K}_-(u)$ satisfying equation (21), one can find that

$$\mathcal{T}(u)=T(u)\mathcal{K}_-(u)T^{-1}(-u) \tag{27}$$

is a solution of equation (21). In the proof of this, the relations $L_j(u)L_k(v)=L_k(v)L_j(u)$, $j\neq k$ have been used. In the quantum inverse scattering method, the Hamiltonian of a system can be given by means of the transfer matrix. For the case of the open chain, we define the transfer matrix as

$$t(u)\doteq \operatorname{Tr}\mathcal{K}_+(u)\mathcal{T}_-(u). \tag{28}$$

By a suitable generalization of Sklyanin's arguments [7], it now follows that the $t(u)$ forms a commutative family

$$[t(u),t(v)]=0. \tag{29}$$

In order to prove equation (29), one can use equation (28) to rewrite

$$\begin{aligned} t(u)t(v)&=\operatorname{Tr}\mathcal{K}_+(u)\mathcal{T}_-(u)\operatorname{Tr}\mathcal{K}_+(v)\mathcal{T}_-(v)\\ &=\operatorname{Tr}_{12}\overset{1}{\mathcal{K}}_+(u)\overset{1}{\mathcal{T}}_-(u)\overset{2}{\mathcal{K}}_+(v)\overset{2}{\mathcal{T}}_-(v)\\ &=\operatorname{Tr}_{12}\{\overset{1}{\mathcal{K}}{}_+^{h_1}(u)\overset{1}{\mathcal{T}}{}_-^{h_1}(u)\overset{2}{\mathcal{K}}{}_+^{h_2}(v)\overset{2}{\mathcal{T}}{}_-^{h_2}(v)\}^{c_1c_2}\end{aligned}$$

where c_i means complex conjugation in ith space. Now one can insert four R matrices using equations (10) and (11) and use the fact that $\mathcal{T}_-$ and $\mathcal{K}_+$ satisfy the equations (21) and (22) to change the order of $\mathcal{T}_-$ and $\mathcal{K}_+$:

$$\begin{aligned}\ldots&=\operatorname{Tr}_{12}\left\{\overset{1}{\mathcal{K}}{}_+^{h_1}(u)(R^{h_1}(-u_+-nw)R^{h_2}(u_+))^{h_2}\overset{2}{\mathcal{K}}{}_+^{h_2}(v)\overset{1}{\mathcal{T}}{}_-^{h_1}(u)\overset{2}{\mathcal{T}}{}_-^{h_2}(v)\frac{1}{\tilde{\rho}(u_+,w)}\right\}^{c_1c_2}\\ &=\operatorname{Tr}_{12}\left\{(\overset{1}{\mathcal{K}}{}_+^{h_1}(u)R^{h_1h_2}(-u_+-nw)\overset{2}{\mathcal{K}}{}_+^{h_2}(v))\frac{1}{\rho(-u_-)\tilde{\rho}(u_+,w)}\right.\\ &\quad\left.\times(\overset{1}{\mathcal{T}}_-(u)R^{h_1h_2}(u_+)\overset{2}{\mathcal{T}}_-(v))^{h_1h_2}R^{h_1h_2}(u_-)R(-u_-)\right\}^{c_1c_2}\\ &=\operatorname{Tr}_{12}\left\{(R(-u_-)\overset{1}{\mathcal{K}}{}_+^{h_1}(u)R^{h_1h_2}(-u_+-nw)\overset{2}{\mathcal{K}}{}_+^{h_2}(v))\frac{1}{\rho(-u_-)\tilde{\rho}(u_+,w)}\right.\\ &\quad\left.\times(R(u_-)\overset{1}{\mathcal{T}}_-(u)R^{h_1h_2}(u_+)\overset{2}{\mathcal{T}}_-(v))^{h_1h_2}\right\}^{c_1c_2}\\ &=t(v)t(u).\end{aligned}$$

In the last step, we omit the calculation similar to second and third steps.

The quantum space, acted on by the operator $S^{(j)}$, is isomorphic to the auxiliary space and, furthermore, the operator $L_j(u)$ coincides with the matrix $R(u)$ on the direct product space of the quantum and auxiliary spaces, i.e.

$$L_j(u) = R_{0j}(u). \tag{30}$$

We know from proposition 1 in [11] that if $R_{jk}(u)$ is normalized, the value of it at $u=0$ is the permutation operator. Differentiating $t(u)$ with respect to u at $u=0$, one can find the Hamiltonian of the open chain

$$H = \sum_{j=1}^{N-1} H_{j,j+1} + \tfrac{1}{2}\overset{1}{\mathscr{K}}{}'_- + \frac{\mathrm{Tr}_0 \overset{0}{\mathscr{K}}_+(0) H_{0,N}}{\mathrm{Tr}\, \mathscr{K}_+(0)} \tag{31}$$

where

$$H_{j,j+1} = P_{jj+1} R'_{jj+1}(u)\big|_{u=0}. \tag{32}$$

Substituting (12), (23) and $\mathscr{K}_\pm$ into (31), we obtain the Hamiltonian of the Belavin model with independent boundary conditions

$$H = \frac{1}{n^2} \sum_{j=1}^{N-1} \sum_{\gamma,\beta \in Z_n^2} W'_\beta(0) \omega^{\langle\gamma,\beta\rangle + \gamma_1\gamma_2} S_\gamma^{(j)} S_{-\gamma}^{(j+1)} + \tfrac{1}{2} \sum_{\gamma,\beta \in Z_n^2} W'_{2\beta}(0) \omega^{2\langle\gamma,\beta\rangle + 2\gamma_1\gamma_2} S_{2\gamma}^{(1)}$$
$$+ \frac{1}{\sum_{\alpha \in Z_n^2} W_{2\alpha}(-nw/2)}$$
$$\times \sum_{\alpha,\beta,\gamma,\rho \in Z_n^2} \left\{ \omega^{2\alpha_1\alpha_2 - 2\beta_1\beta_2 + \langle\gamma,\rho\rangle + \gamma_1\gamma_2} W_{2\beta}\left(-\frac{nw}{2}\right) W'_\rho(0) \delta_{2(\alpha-\beta),\gamma}^{\mathrm{mod}\, n} S_\gamma^{(N)} \right\} \tag{33}$$

where $\langle\gamma, \beta\rangle = \gamma_1\beta_2 - \gamma_2\beta_1$. It is worth pointing out that the Hamiltonian (33) generally is not Hermitian as in the case of periodic boundary conditions because the parameters τ and $w(\eta)$ are complex variables. For periodic case, the exact solution of the Z_n Belavin model was found by Hou *et al* [3] and the eigenvalue of the Hamiltonian is not real. Generally, when the argument τ in the theta function approaches $\mathrm{i}\infty$ and w is well defined in proper field, the Hamiltonian is Hermitian. This is hinted at by the works of Hou *et al* [12], Pasquier and Saleur [8]. We will discuss this problem in detail and find the solution of the Hamiltonian (33) in a future paper.

In conclusion, we have generalized Sklyanin's formalism for constructing integrable open chains to the case of an$R$-matrix satisfying (9)–(11). As a direct application of our extension, we have constructed the Hamiltonian of the open chain corresponding to the Belavin model.

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References

[1] Baxter R J 1982 *Exactly Solvable Model in Statistical Mechanics* (London: Academic)
[2] Belavin A A 1980 *Nucl. Phys.* B **180** 109
[3] Hou B Y, Yan M L and Zhou Y K 1989 *Nucl. Phys.* B **324** 715
[4] Kulish P P, Reshetikhin N Yu and Sklyanin E K 1981 *Lett. Math. Phys.* **3** 393
[5] Date E, Jimbo M, Kuniba A, Miwa T and Okado M 1987 *Nucl. Phys.* B **290** 231
[6] Takhtajan L A and Faddeev L D 1979 *Russ. Math. Surv.* **34**
[7] Sklyanin E K 1988 *J. Phys. A: Math. Gen.* **21** 2375
[8] Pasquier N and Saleur H 1990 *Nucl. Phys.* B **330** 523
[9] Mezincescu L and Nepomechie R I N 1991 *J. Phys. A: Math. Gen.* **24** L17: 1991 *Mod. Phys. Lett.* **6** A 2497
[10] Bazhanov V V 1987 *Comm. Math. Phys.* **113** 471
[11] Yue R H and Chen Y X *Preprint* AS-ITP-92-20
[12] Hou B Y, Shi K J and Yang Z X *Preprint* IMP-NWU-91
[13] Hou B Y and Wei H 1989 *J. Math. Phys.* **30** 2750
[14] Cherednik I V 1984 *Theor. Math. Phys.* **61** 911; 1983 **17** 77